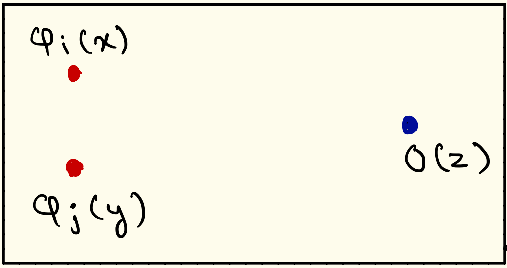
Operator Product Expansion

Recall that a scaling operator φ_i satisfies
$\langle \varphi_i(r)\,\varphi_i(0)\rangle \sim \dfrac{1}{r^{2x_i}}$ where x_i is the scaling dimension of φ_i. Further, each φ_i can be expressed as a linear combination of lattice operators S_a

$$\varphi_i = \sum_a c_{ia}\, S_a .$$

Now consider a correlation f^n of the form $\langle \varphi_i(x)\,\varphi_j(y)\,O(z)\rangle$ where O is some generic operator. The case which will be of interest to us is $|\vec{x}-\vec{y}| \ll |\vec{x}-\vec{z}|, |\vec{y}-\vec{z}|$ i.e. φ_i and φ_j are close to each other relative to O.



Expressing $\varphi_i(x)$ and $\varphi_j(y)$ in terms of microscopic operators S_a, the above correlation f^n is,

$$\sum_{ab} \langle c_{ia}\, S_a(x)\, c_{jb}\, S_b(y)\, O(z)\rangle.$$

The main intuition one employs is that due to $|\vec{x}-\vec{y}| \ll |\vec{x}-\vec{z}|, |\vec{y}-\vec{z}|$, one may want to replace in this expression $\sum_{ab} c_{ia}\, c_{jb}\, S_a(x) S_b(y)$

$\simeq \sum_k C_{ijk}(x-y)\, \varphi_k\left(\frac{x+y}{2}\right)$ since this operator appears local from the point of view of $O(z)$.

Essentially, one is inverting the relation of kind $\varphi_i \doteq \sum_a c_{ia} S_a$ for the operator $\sum_{ab} c_{ia}\, c_{jb}\, S_a(x)\, S_b(y)$.

This replacement is valid <u>only</u> when calculating expectation values such as $\langle \varphi_i(x)\, \varphi_j(y)\, O(z) \rangle$ with $|\vec{x}-\vec{y}| \ll |\vec{x}-\vec{z}|, |\vec{y}-\vec{z}|$.

Therefore, one writes,

$$\varphi_i(\vec{r}_1)\, \varphi_j(\vec{r}_2) = \sum_k C_{ijk}(r_1-r_2)\, \phi_k\left(\frac{r_1+r_2}{2}\right)$$

The dependence of $C_{ijk}(r_1-r_2)$ on r_1-r_2 can be obtained via conformal invariance,

$$C_{ijk}(r_1-r_2) = \frac{c_{ijk}}{|r_1-r_2|^{x_i+x_j-x_k}}$$

One assumption we have used here is that φ_i's transform as spin-zero operators under rotations.

Another point to note is that one would also involve <u>derivatives</u> of scaling operators when expressing $\varphi_i(r_1)\,\varphi_j(r_2)$ i.e.

$$\varphi_i(r_1)\,\varphi_j(r_2) \sim \sum_k C_{ijk}(r_1-r_2)\,\varphi_k + \sum_k (\vec{r}_1-\vec{r}_2)\cdot\vec{\nabla}\varphi_k\, C_{ijk}(r_1-r_2) + \text{higher derivatives.}$$

Although these extra terms exist, we will drop them since in a system without boundary they will integrate to zero in the Hamiltonian.

The numbers C_{ijk} are called OPE coefficients, and their value depends on the normalization convention for correlation functions. One way to fix normalisation is by requiring $\langle \varphi_i(r)\,\varphi_i(0)\rangle = \dfrac{1}{r^{2x_i}}$. Having chosen such a convention, C_{ijk} are <u>universal</u> in the same sense as x_i.

RG flow using OPE

Let's consider a system, which is initially at a RG fixed point equipped with scaling operators $\{\phi_i\}$, to be perturbed by a combination of scaling operators:

$$H = \underbrace{H^*}_{\text{Fixed-point Hamiltonian}} + \underbrace{\sum_i g_i \, a^{x_i} \sum_r \phi_i(r)}_{\text{perturbation}}$$

The factors of a^{x_i} ensure that the couplings g_i are dimensionless.

If the perturbation is relevant, one will flow to a different fixed point, with a new set of scaling operators. However, as we now discuss, that the <u>perturbative</u> RG flow, in the vicinity of the original fixed point $(\equiv H^*)$ can be obtained solely in terms of the OPE coefficients c_{ijk}.

The main tool we will employ is that the partition f^n is invariant under an RG transformation.

Consider the partition f^n :

$$Z(a) = \mathrm{Tr}\, e^{-H}$$
$$= \mathrm{Tr}\, e^{-[H^* + \sum_i g_i a^{x_i} \varphi_i(r)]}$$
$$= \mathrm{Tr}\, e^{-[H^* + \sum_i g_i \int \frac{d^d r}{a^{d-x_i}} \varphi_i(r)]}$$

where we have used the convention $\sum_r = \int \frac{d^d r}{a^d}$

on dimensional grounds.

Expanding upto quadratic order in $\{g_i\}$:

$$Z(a) = \mathrm{Tr}\, e^{-H^*} \Big[1 - \sum_i g_i \int \frac{d^d r}{a^{d-x_i}} \varphi_i(r)$$
$$+ \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^d r_1\, d^d r_2}{a^{2d - x_i - x_j}} \varphi_i(r_1) \varphi_j(r_2) \Big] + \dots$$

$$= Z^* \Big[1 - \sum_i g_i \int \frac{d^d r}{a^{d-x_i}} \langle \varphi_i(r) \rangle$$
$$+ \frac{1}{2} \sum_{ij} g_i g_j \int \frac{d^d r_1\, d^d r_2}{a^{2d-x_i-x_j}} \langle \varphi_i(r_1) \varphi_j(r_2) \rangle$$
$$+ \dots \Big]$$

where $Z^* = \mathrm{Tr}\, e^{-H^*}$.

Under the rescaling $a \to ba = (1+\delta l)a$, the partition f^n should be invariant, $Z(a) = Z(ba)$. The dependence on 'a' appears in three ways:

- explicit dependence on 'a' in factors of a^{d-x_i} etc.
- In the second term when $r_1 \to r_2$, one will encounter divergences (since $\langle \phi_i(r)\phi_j(0)\rangle \sim \frac{1}{r^{x_i+x_j}}$). To avoid these divergences, we need to restrict $|r_1 - r_2| > a$.
- Through dependence on $\frac{L}{a}$ where L is the total system size. This dependence will be taken into account through the usual finite-size scaling arguments we discussed earlier and will not an important role in the following. Therefore, we will not discuss it here.

Explicitly,

$$Z(a) = Z^* \Big[1 - \sum_i g_i \int \frac{d^d r}{a^{d-x_i}} \langle \phi_i(r) \rangle$$

$$+ \frac{1}{2} \sum_{ij} g_i g_j \int_{|r_1-r_2|>a} \frac{d^d r_1\, d^d r_2}{a^{2d-x_i-x_j}} \langle \phi_i(r_1)\, \phi_j(r_2) \rangle \Big]$$

After rescaling $a \to ba$,

$Z(ba)$

$$= Z^* \Big[1 - \sum_i g_i \int \frac{d^d r}{(ba)^{d-x_i}} \langle \varphi_i(r) \rangle$$

$$+ \frac{1}{2} \sum_{ij} g_i g_j \int_{|r_1 - r_2| > ba} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \langle \varphi_i(r_1) \, \varphi_j(r_2) \rangle \Big]$$

Let's divide the second term into two parts:

$$\frac{1}{2} \sum_{ij} g_i g_j \int_{|r_1 - r_2| > a} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \langle \varphi_i(r_1) \, \varphi_j(r_2) \rangle$$

$$-\frac{1}{2} \sum_{ij} g_i g_j \int_{ba > |r_1 - r_2| > a} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \langle \varphi_i(r_1) \, \varphi_j(r_2) \rangle$$

The second of these terms may be simplified using OPE :

$$-\frac{1}{2} \sum_{ij} g_i g_j \int_{ba > |r_1 - r_2| > a} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \langle \varphi_i(r_1) \, \varphi_j(r_2) \rangle$$

$$= -\frac{1}{2} \sum_{ij} g_i g_j \int_{ba > |r_1 - r_2| > a} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \frac{c_{ijk} \, \langle \varphi_k(\frac{r_1 + r_2}{2}) \rangle}{|r_1 - r_2|^{x_i + x_j - x_k}}$$

This term may be further simplified by using a new set of coordinates, $\vec{r} = \frac{\vec{r}_1 + \vec{r}_2}{2}$, $\vec{S} = \frac{\vec{r}_1 - \vec{r}_2}{2}$:

$$-\frac{1}{2}\sum_{ij} g_i g_j \int_{ba > |r_1 - r_2| > a} \frac{d^d r_1\, d^d r_2}{(ba)^{2d - x_i - x_j}} \, c_{ijk} \frac{\langle \varphi_k(\frac{r_1+r_2}{2}) \rangle}{|r_1 - r_2|^{x_i + x_j - x_k}} \Big]$$

$$= -\frac{1}{2} g_i g_j \int d^d r \int_{ba > S > a} d^d S \; \frac{c_{ijk}}{(ba)^{2d - x_i - x_j}} \frac{\langle \varphi_k(r) \rangle}{S^{x_i + x_j - x_k}}$$

Since $b = 1 + \delta l \simeq 1$, $\Rightarrow |\vec{S}| \simeq a$. Therefore, one may perform the integral over $\vec{S}$ to rewrite this term as:

$$-\frac{1}{2} g_i g_j \int d^d r \; S_d \, a^d (b-1) \frac{c_{ijk}}{(ba)^{2d - x_i - x_j}} \frac{\langle \varphi_k(r) \rangle}{a^{x_i + x_j - x_k}}$$

where S_d is the surface area of a unit sphere embedded in d-dimensions. The factors of 'a' combine to give $a^{x_k - d}$, as expected on dimensional grounds.

Combining everything,

$$Z(ba) =$$

$$Z^* \Big[1 - \sum g_k \int \frac{d^d r}{(ba)^{d-x_k}} \langle \varphi_k(r) \rangle$$

$$+ \frac{1}{2} \sum_{ij} g_i g_j \int_{|r_1 - r_2| > a} \frac{d^d r_1 \, d^d r_2}{(ba)^{2d - x_i - x_j}} \langle \varphi_i(r_1) \, \varphi_j(r_2) \rangle$$

$$- \frac{1}{2} \sum_{ijk} g_i g_j \int d^d r \, S_d \, \frac{(b-1) \, c_{ijk}}{b^{2d - x_i - x_j}} \frac{\langle \varphi_k(r) \rangle}{a^{d - x_k}} \Big]$$

Our aim is to find a rescaling of $\{g_i\}$ so that $Z(ba) = Z(a)$. We claim that the following transformation does the job:

$$g_k(b) = \underbrace{g_k \, b^{d - x_k}}_{\mathrm{I}} - \underbrace{\frac{1}{2} S_d \sum_{ij} c_{ijk} \, g_i g_j (b-1)}_{\mathrm{II}}$$

The contribution I cancels out all factors of b^{d-x_k} in the denominator, while the II contribution cancels out the third term proportional to $(b-1)$.

Therefore, the RG flow is :

$$\frac{dg_k}{dl} = (d - x_k) g_k - \frac{S_d}{2} \sum_{ij} c_{ijk} g_i g_j$$

Rescaling $g_k \to \frac{2}{S_d} g_k$, one obtains,

$$\boxed{\frac{dg_k}{dl} = y_k g_k - \sum_{ij} c_{ijk} g_i g_j}$$

where $y_k = d - x_k$ is the RG eigenvalue of g_k at the fixed point given by the unperturbed Hamiltonian $\mathcal{H}^*$.

ε - expansion using OPE

As an application of the above derived RG flow using OPE, let's consider perturbing the Gaussian fixed point in $d = 4 - \varepsilon$ dimensions. The fixed point Hamiltonian is

$$H^* = \frac{1}{2} \int d^d r \, (\nabla \varphi)^2 .$$

The perturbation is,

$$\delta H = t \, a^{x_t} \int d^d r \, \varphi^2(r) + u \, a^{x_u} \int d^d r \, \varphi^4(r)$$

where $x_t = d - y_t = d - 2$, $x_u = d - y_u = 2d - 4$

Based on the above derived RG equation,

$$\frac{dt}{dl} = 2t - c_{ttt} t^2 - c_{tut} ut - c_{utt} tu - c_{uut} u^2$$

$$\frac{du}{dl} = (4 - d) u - c_{ttu} u^2 - c_{tuu} ut - c_{utu} tu - c_{uuu} u^2$$

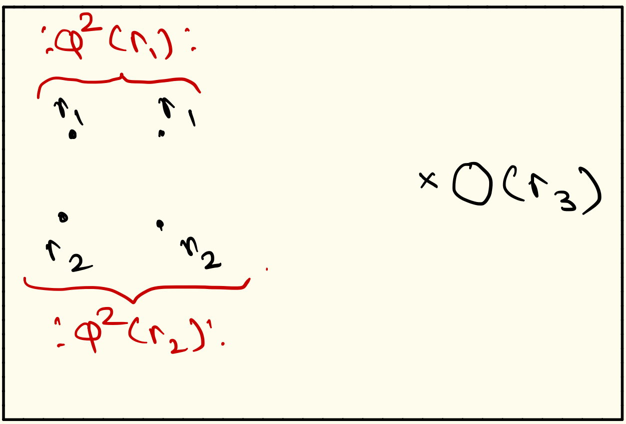
Note that $C_{ijk} = C_{jik}$ since $\langle \varphi_i(x) \varphi_j(0) \rangle$ $= \langle \varphi_j(x) \varphi_i(0) \rangle$ from Poincare invariance.

Therefore, we need C_{ttt}, C_{tut}, C_{uut}, C_{ttu}, C_{tuu}, C_{uuu}, i.e. we need OPE of $\varphi^2(x)\, \varphi^2(0)$, $\varphi^2(x)\, \varphi^4(x)$ and $\varphi^4(x)\, \varphi^4(0)$.

It turns out to be quite convenient to define <u>normal-ordered</u> operators $:O:$ which have the property $\langle :O: \rangle = 0$. For example, $:\varphi^2: = \varphi^2 - \langle \varphi^2 \rangle$. The advantage is that in the OPE calculation below, only the connected diagrams contribute to the normal ordered operators. To obtain $:\varphi^n:$, one way do it recursively by demanding $\langle :\varphi^n: \rangle = 0$ and $\langle :\varphi^n: \, :\varphi^m: \rangle = 0$ when $n \neq m$. For example, consider writing $:\varphi^4: = \varphi^4 + a\, \varphi^2 \langle \varphi^2 \rangle + b \langle \varphi^2 \rangle^2$ where a, b are unknown. Demanding these two conditions, one finds $:\varphi^4: = \varphi^4 - 6\langle\varphi^2\rangle\varphi^2 + \langle\varphi^4\rangle$. This is slightly different than Cardy's convention $:\varphi^4: = \varphi^4 - 3\langle\varphi^2\rangle\varphi^2$ which of course satisfies $\langle :\varphi^4: \rangle = 0$ but doesn't satisfy $\langle :\varphi^4:(x) :\varphi^2:(y)\rangle = 0$. That is OK as it doesn't affect the property of only connected diagrams contributing to OPE, so one may use either definitions.

Let's consider a few examples to illustrate OPE calculation:

(i) $:\varphi^2(r_1): \; :\varphi^2(r_2):$



To calculate OPE, we imagine an arbitrary operator $O(r_3)$ far away from r_1, r_2 and consider calculating $\langle :\varphi^2(r_1)::\varphi^2(r_2): O(r_3)\rangle$

This correlation f^n may be calculated using Wick's thm (since the theory is Gaussian)

That is, one considers all possible ways of contracting operators.

For example, for operators that are not normal ordered:

$$\langle \varphi^2(r_1)\ \varphi^2(r_2) \rangle = \langle \varphi(r_1)\, \varphi(r_1)\ \varphi(r_2)\, \varphi(r_2) \rangle$$

$$= \langle \varphi(r_1)\, \varphi(r_1) \rangle \langle \varphi(r_2)\ \varphi(r_2) \rangle$$
$$+ 2 \langle \varphi(r_1)\, \varphi(r_2) \rangle \langle \varphi(r_1)\, \varphi(r_2) \rangle$$

In contrast, for normal ordered operators,

$$\langle :\varphi^2(r_1):\ :\varphi^2(r_2): \rangle$$

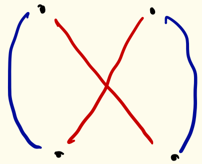
$$= 2\ \langle \varphi(r_1)\ \varphi(r_2) \rangle\ \langle \varphi(r_1)\ \varphi(r_2) \rangle$$

That is, the terms where contractions occur at the same point in space do not contribute.

Returning to $\langle :\varphi^2(r_1): :\varphi^2(r_2): \; O(r_3)\rangle$, there are three set of terms:

(a) Operators $:\varphi^2(r_1):$ and $:\varphi^2(r_2):$ contract with each other completely:

$\varphi(r_1)$ $\varphi(r_1)$



$\varphi(r_2)$ $\varphi(r_2)$

[Two options]

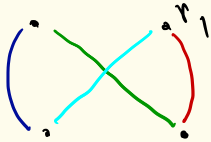
In this case, from a far away point, their insertion appears as identity operator, i.e. they contribute to the above expectation value,

$$\langle :\varphi^2(r_1): :\varphi^2(r_2):\rangle\langle O(r_3)\rangle$$

$$= \frac{2}{|r_1-r_2|^{2d-4}}\langle O(r_3)\rangle \quad + \text{ other contributions.}$$

(b) Only two operators at r_1, r_2 contract with each other and the rest contract with $O(r_3)$:

r_1 r_1



r_2 r_2

[four options]

$$\langle :\varphi^2(r_1): :\varphi^2(r_2): \; O(r_3)\rangle \simeq 4\,\langle\varphi(r_1)\,\varphi(r_2)\rangle \langle :\varphi^2(r_1): \; O(r_3)\rangle$$

$$= \frac{4}{|r_1 - r_2|^{d-2}} \langle :\varphi^2(r_1): \; O(r_3) \rangle$$

$$+ \text{ other contributions.}$$

where we have replaced $\varphi(r_1)\,\varphi(r_2)$ with $\varphi^2(r_1)$ since from a distant point

$\varphi(r_1)\,\varphi(r_2) \simeq \varphi^2(r_1)$ + higher derivatives that don't contribute to H.

Thus, this contribution appears as an insertion of operator $:\varphi^2(r_1):$ at $r_1 \simeq r_2$.

(c) The last possibility is that none of the operators at r_1, r_2 are contracted with each other, so that $:\varphi^2(r_1): \, :\varphi^2(r_2):$ appears as $:\varphi^4(r_1):$ from a distant point.

Combining everything,

$$:\varphi^2(r_1)::\varphi^2(r_2):$$

$$= \frac{2}{|r_1-r_2|^{2d-4}} + \frac{4}{|r_1-r_2|^{d-2}} :\varphi^2(r_1): + :\varphi^4(r_1):$$

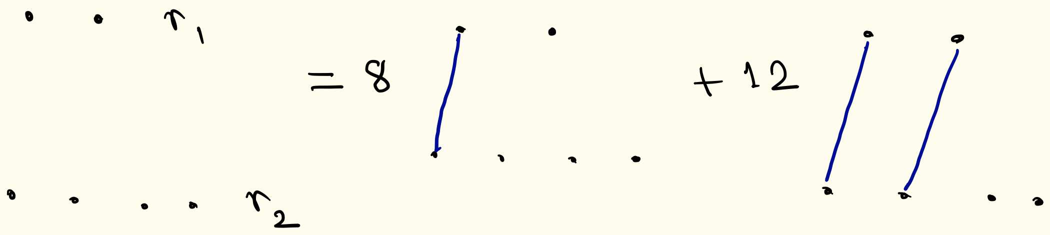
Or, in terms of OPE equation,

$$\varphi^2 \cdot \varphi^2 = 2 + 4\varphi^2 + \varphi^4$$

$$\Rightarrow \quad c_{ttt} = 4 , \quad c_{ttu} = 1 .$$

Let's consider another example,

$$:\varphi^2(r_1)::\varphi^4(r_2):$$



$$= 8 :\varphi^4: + 12 :\varphi^2:$$

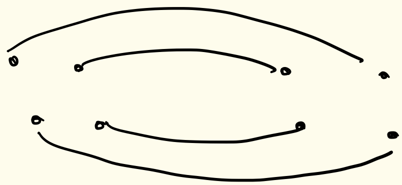
$$\Rightarrow \quad c_{tut} = 12 , \quad c_{tuu} = 8 .$$

Finally consider,

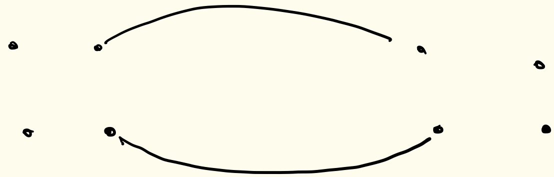
$:\varphi^4(r_1)::\varphi^4(r_2):$

r_1 $\qquad\qquad\qquad$ r_2

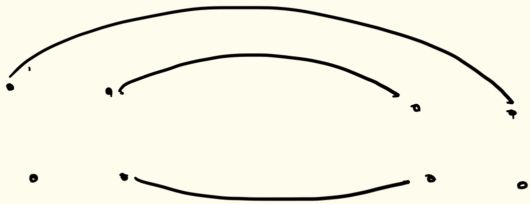
$= 24$
$(= 4!)$



$+ \quad 72$
$(= 6 \times 6 \times 2)$



$+ \quad 96$
$(= 4 \times 4 \times 6)$



$$= 24 + 72 :\varphi^4: + 96 :\varphi^2:$$

$\Rightarrow C_{uu1} = 24, \quad C_{uut} = 96, \quad C_{uuu} = 72.$

Using the above obtained OPE coeffs,

$$\frac{dt}{dl} = 2t - 4t^2 - 24ut - 96u^2$$

$$\frac{du}{dl} = (4-d)u - 16ut - 72u^2.$$

When $4-d=\varepsilon$ is small, one obtains,

$t^* = O(\varepsilon^2) \equiv 0$ at $O(\varepsilon)$,

$$u^* = \frac{\varepsilon}{72}$$

Linearizing around the fixed point,

$$\frac{dt}{dl} = (2 - 24u^*)t$$

$$= \left(2 - \frac{\varepsilon}{3}\right)t$$

$\Rightarrow \quad y_t = 2 - \frac{\varepsilon}{3} \quad \Rightarrow \quad \nu = \frac{1}{2} + \frac{\varepsilon}{12}$

in agreement with our result obtained earlier using Wilson RG.